PHYS 731 – Quantum Computing Architectures and Algorithms

Practice-Problem Set – Spring Semester 2026

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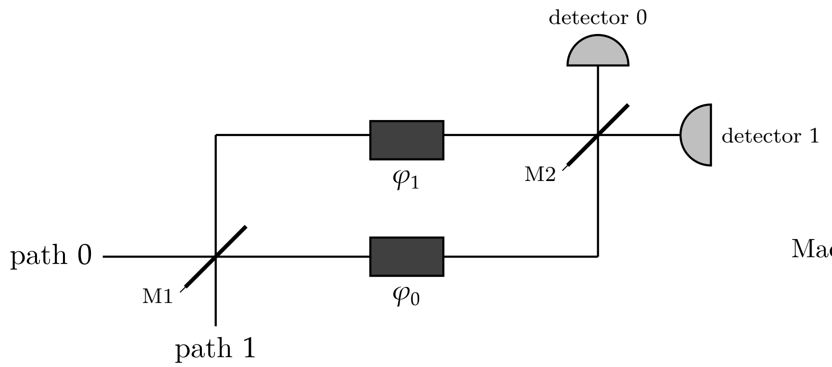
1. Mach–Zehnder Interferometer (*or all there is to a quantum computation*)

   In the Mach–Zehnder interferometer, the mirrors (M1,2) represent a 1/2 probability for a photon to continue straight through them or to be deflected. If such a mirror deflects to the right, the photon accumulates a phase, e.g., $|1\rangle \to e^{i\pi}|0\rangle$, while changing direction at M1 coming in as $|0\rangle$, $|0\rangle \to |1\rangle$. M1 sends any incoming photon onto either of two paths along which they pick up phases φ_0 and φ_1 respectively. M2 recombines the two paths, in other words, undoes the splitting or whatever has been done to represent the original photon state. Thus, in each detector, a linear combination of photons having taken the two paths are detected.

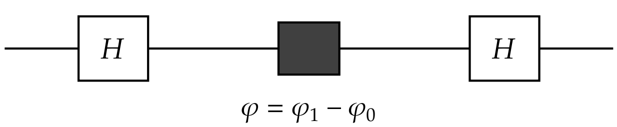


Mach–Zehnder Interferometer

For the following, consider photons entering the interferometer on path 1 with phase zero. For the general photon state, use the following notation: $e^{i(\texttt{phase})}|\texttt{path}\rangle$.

(a) What is the probability of detecting the photon in each detector?

(b) Does that probability depend on the fact that you sent in photons on path 1 instead of path 0?



1-Qbit quantum-circuit representation

(a) Explain how the Mach–Zehnder interferometer can be represented by the quantum circuit above.

Hint: If the Qbit is initially in state $|0\rangle$, what are the probabilities to measure it after the two Hadarmard gates and the phase gate in one of the two computational basis states?

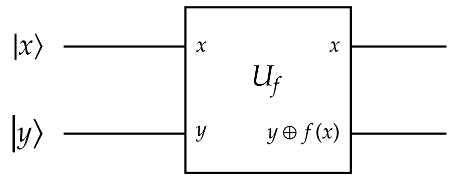
(b) For a general phase φ, sketch a circuit which represents the black box in terms of T and H gates.

ECCE: The essence of any quantum computation is included in these identical experiments:

1. Prepare an input state which encodes as a linear combination the information you want to process.
2. Encode the result of the computation on each of these terms in their respective phases.
3. Undo the initial decomposition, i.e., recombine particles having taken different paths, i.e., measure in the same basis as the initial state was prepared.

2. Quantum Parallelism

Consider the so-called *f-controlled-NOT* $(f - c - N)$ 2-Qbit circuit below. Without specifying the gate structure within the box, its effect is to pass the upper wire unchanged but use its value x as the argument of a Boolean function $f : \{0,1\} \rightarrow \{0,1\}$ whose result is added (mod 2) to the lower wire's input y.



$$|x\rangle |y\rangle \overset{f-c-N}{\longrightarrow} |x\rangle |y \oplus f(x)\rangle$$

(a) In order to implement a controlled-NOT gate, what should f be?

(b) Note that f accepts quantum states as input. In order to encode $f(0)$ and $f(1)$ in the output, what input state should we use, and how would you modify the circuit such that it yields this input from the initial state $|x\rangle |y\rangle = |0\rangle |0\rangle$?

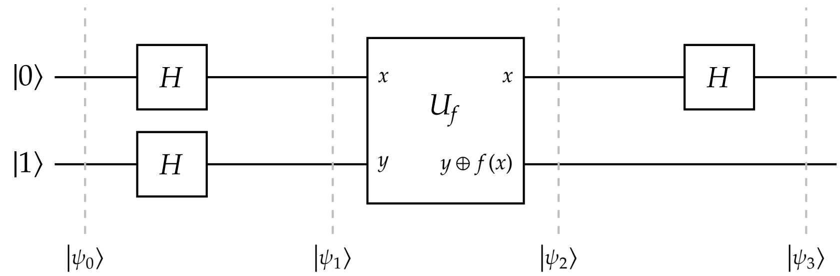
(c) Without adding y (mod 2) to the output, would the circuit represent a valid quantum operation? Why or why not?

(d) Now you want to exploit this quantum feature in order to have with a single "call" to $f : \{0,1\}^n \rightarrow \{0,1\}$ its results for **all** possible inputs $\{0,1\}^n$ for an n-Qbit input present in the output.

Hint: You can use the $f - c - N$ circuit as a building block which accepts the n-Qbit input $|x\rangle$, and the only problem you have is to prepare a superposition of all possible inputs.

3. Deutsch–Jozsa Algorithm

The circuit below is a specific variant of the one introduced above in Question 2.



(a) What are the states $|\psi_0\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$?

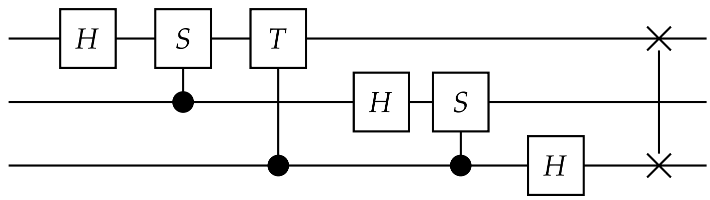
(b) Imagine yourself a detective who wants to figure out something about the unknown function f. But you are lazy, and you want to "query" f as few times as possible. First, think, and explain how many different f's there are.

(c) So there are constant (0 or 1 regardless of input) and balanced (as many 0's as 1's in the output) functions. For each type, what is the state $|\psi_2\rangle$?

(d) What purpose does the final H gate serve? What is the state $|\psi_3\rangle$ for each type of f?

4. <u>3-Qbit Quantum Fourier Transform</u>

Consider the following as one way to encode information of interest in another linear combination than the one used above for the Deutsch(-Jozsa) algorithm. The S and T gates are the single-Qbit phase gates defined as $S = |0\rangle\langle 0| + i\,|1\rangle\langle 1|$ and $T = |0\rangle\langle 0| + e^{i\pi/4}\,|1\rangle\langle 1|$, respectively, and H is the Hadamard gate. The last element of the circuit swaps/interchanges Qbits 1 and 3.



(a) Give the unitary matrix representation of each of the gates in the circuit (except for the last element).

(b) Find the matrix representation of this circuit. Express your answer using the single parameter $\omega = e^{i\pi/4}$.

(c) Express the action of this circuit on the computational basis

$$\left|ijk\right\rangle \equiv \left|i\cdot 2^0 + j\cdot 2^1 + k\cdot 2^2\right\rangle \equiv |n\rangle \quad \text{with } n \in \{0,1,\ldots,7\} \quad \text{and} \quad i,j,k \in \{0,1\}$$

and hence

$$|n\rangle \overset{\texttt{circuit}}{\longrightarrow} \frac{1}{\sqrt{8}} \sum_{l=0}^{7} \left(e^{2\pi i/8}\right)^{n\cdot l} |l\rangle \quad . \qquad \text{(3-Qbit FT)}$$

(d) Express the sum in Eq. (3-Qbit FT) as a product of three sums, each of which representing a single-Qbit state:

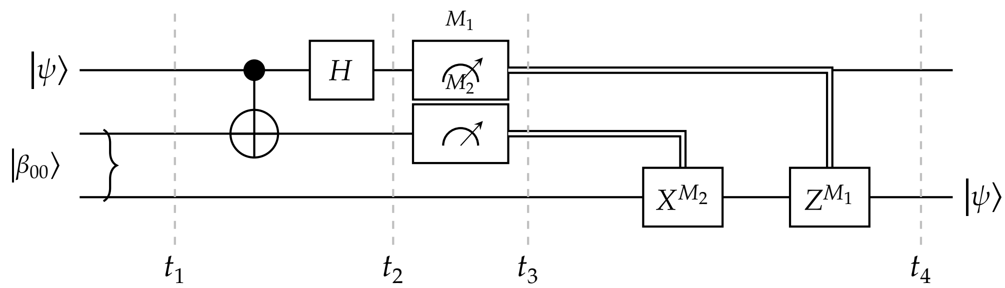
$$|n\rangle \overset{\texttt{circuit}}{\longrightarrow} \left(|0\rangle + y_0\,|1\rangle\right) \otimes \left(|0\rangle + y_1\,|1\rangle\right) \otimes \left(|0\rangle + y_2\,|1\rangle\right) \quad .$$

Hint: Use the binary representation of l and n.

(e) Foresee how to generalize this transformation to an n-Qbit input.

5. Quantum Teleportation

Can Ifora transmit an unknown Qbit-state $|\psi\rangle$ to Imronbek without sending him the physical particle but only the values of two classical bits?



Ifora and Imronbek meet at a lab where they prepare two Qbits (the lower two wires) in the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Time passes, and Ifora moves to Fergana and Imronbek has to work in Nukus. Each of them takes one of the Bell-pair Qbits along the way.

(a) What is the state of all three Qbits at t_1, just before Ifora subjects her part of the entangled pair to a NOT gate controlled by the Qbit she finds so beautiful that she wants to send it to Imronbek? What state does she own at t_2 (use $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ for the unknown “beauty”)?

(b) Since atoms are fragile, Ifora measures her two Qbits between t_2 and t_3 in the computational basis, and sends the result to Imronbek via *Telegram* (double lines in the sketch). Choose one possible measurement result and find the state of Imronbek’s Qbit at t_4.

(c) Does this amazing result depend upon your choice of the result and/or on the initial choice to entangle in the Bell state $|\beta_{00}\rangle$?

6. 3-step classification

In each of the above questions, you investigated a specific quantum circuit. For each of those, identify the three steps given in problem 1, or understand the algorithm as one of those three essential steps: preparation, computation, interference.