

Quantum Mechanics I

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Syllabus

Course structure

15 weeks, two 100-minute lectures per week. For each week, 6 problems (2+2+2 for three levels of difficulty), a list of suggested reading (including original and experimental sources), and a 1-page¹ summary sheet are provided. The midterm exam contributes 30%, the final exam 70%. The exam questions will be variants of the weekly problems.

Course content (tentative)

Wk	Theme	lectures: keywords
I	systems, states, measurement	$\aleph_I$: the need for a non-classical description of physical systems (listing by Dirac); the spin/polarization experiment as the generic example; summary of vector spaces and complex numbers; Dirac notation and how (dual) vectors, inner products, and outer products are represented; polarization of a photon as a “classical” example of a two-level, quantum system; $\beth_I$: relation between states and vectors; spin states in Dirac notation, and how the observation of the spin/polarization measurement is described in this framework; postulates of quantum mechanics including the operator-measurement correspondence (matrix representation, eigenvalues); illustration with spin operators;
II	dynamics, symmetry	$\aleph_{II}$: discretized amplitude hopping; continuum limit; Schrödinger equation; wave function as amplitude for “acts as if”; position and momentum representations. $\beth_{II}$: unitary transformation; Hermitian generator; translation, rotation, time; conservation as $[A, U] = 0$; stationary states; eigenvalue problem.
III	harmonic oscillator	$\aleph_{III}$: algebraic solution; $a, a^\dagger$; ladder; zero point; representation change as coordinate change. $\beth_{III}$: coherent states; quadratures; phase space; the state that follows a classical trajectory, and why it is the exception. First promise redeemed in lecture XXVI.

¹Named after Paulo Bedaque who envisioned the perfect presentation in a 1-slide format to me.

Wk	Theme	lectures: keywords
IV	angular momentum I	$\aleph_{IV}$: rotation generators; $\mathfrak{su}(2)$ algebra; ladder operators; spectrum from the algebra alone. $\beth_{IV}$: matrix representations; spin- $\frac{1}{2}$ and the qubit identified; 2π rotation and the sign.
V	angular momentum II	$\aleph_V$: orbital angular momentum; spherical harmonics; parity. $\beth_V$: rotation matrices; Wigner d ; irreducible tensors; Wigner–Eckart.
VI	central forces	$\aleph_{VI}$: radial equation; centrifugal barrier; bound states; degeneracy. $\beth_{VI}$: hydrogen; scaling; accidental degeneracy and its symmetry origin.
VII	approximation	$\aleph_{VII}$: time-independent perturbation theory, non-degenerate and degenerate; variational principle; Hellmann–Feynman. $\beth_{VII}$: time-dependent perturbation theory; Fermi golden rule; adiabatic theorem; Berry phase.
VIII	magnetic fields	$\aleph_{VIII}$: minimal coupling; gauge; Landau levels. $\beth_{VIII}$: Zeeman (now with perturbation theory available); Aharonov–Bohm; geometric phase revisited.
IX	two-level dynamics	$\aleph_{IX}$: spin precession; Larmor; Rabi oscillation; rotating frame. $\beth_{IX}$: pulses as gates; Bloch-sphere trajectories; NMR as the original quantum control.
X	coupling of angular momenta	$\aleph_X$: addition; Clebsch–Gordan; singlet and triplet. $\beth_X$: fine and hyperfine structure; the 21 cm line; atomic clocks.
XI	mixed states	$\aleph_{XI}$: density matrix; purity; Bloch ball; reduced state; partial trace. $\beth_{XI}$: von Neumann’s measurement chain; where the cut is placed; decoherence and the preferred basis; Kraus maps and the Lindblad equation in outline.
XII	correlations	$\aleph_{XII}$: entanglement; Schmidt decomposition; monogamy; no-cloning. $\beth_{XII}$: Bell inequality; CHSH; loopholes; what locality and realism each cost; teleportation.
XIII	transformation theory	$\aleph_{XIII}$: representation as a choice of basis; continuous spectra; transformation functions $\langle x p \rangle$; unitary equivalence; Schrödinger, Heisenberg and interaction pictures as one content in three dresses. $\beth_{XIII}$: quantum conditions; Poisson bracket to commutator; canonical quantization and where it is ambiguous; Ehrenfest’s theorem; coherent states revisited; in what sense a limit is classical.
XIV	symmetry and conservation	$\aleph_{XIV}$: symmetry groups; generators and Lie algebra; translation, rotation, Galilean boost, and the conserved quantity belonging to each; degeneracy as a group statement; Runge–Lenz and the hydrogen degeneracy of lecture XII explained. $\beth_{XIV}$: Wigner’s theorem, symmetries are unitary or antiunitary; parity; time reversal and its antiunitarity; Kramers degeneracy; permutation symmetry stated as a postulate and deliberately not developed.
XV	action and path integral	$\aleph_{XV}$: Dirac’s 1933 paper; Feynman’s sum over paths; the propagator; the free particle done completely; stationary phase and the emergence of the classical trajectory.

Wk	Theme	lectures: keywords
		$\beth_{XV}$: the action principle; Noether and Schwinger; symmetry to conservation law inside the path integral; Aharonov–Bohm as a phase around a loop; what the advanced course opens with.

Scriptum

[N_I] The essence of quantum behavior

A description in which objects carry their properties of which they can be asked at any time by pertinent measurements anytime is not how nature works. There are, however, objects which always *answer* certain questions with a definite answer, and we want to call those objects “large”. In contrast, it was found very early that there is the other kind, the “small” system which does reply in a seemingly random way. For the latter, we want to devise a quantitative description. An all encompassing example is the behavior of light of extremely low intensity. The sought-after description must account for:

- Indivisibility,
- polarization,
- interference,
- superposition, and
- indeterminacy.

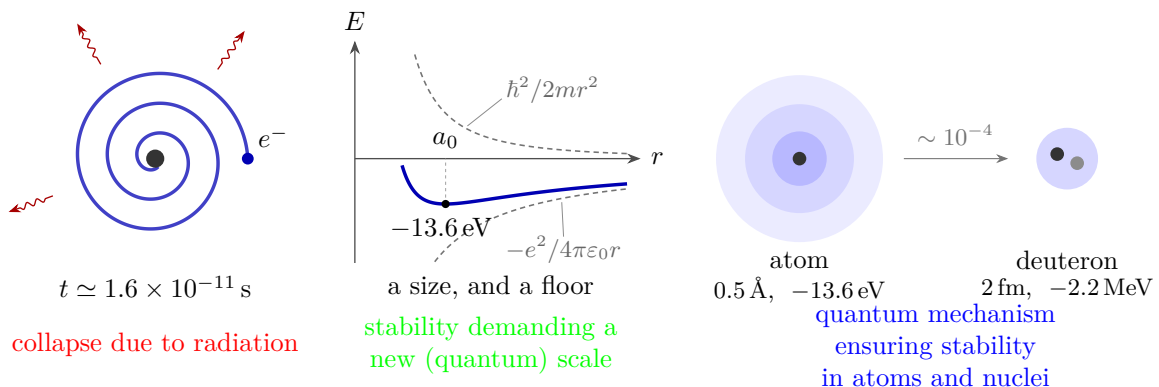
And, an explanation of Rutherford’s scattering (1911) which goes beyond the “unstable” one suggesting the atom as a planetary system of electrons orbiting a heavy nucleus by means of electromagnetic attraction.

Larmor tells us that any accelerated charge radiates away power, and a classical lifetime estimate would predict the electrons to spiral onto the nucleus in a very short time.

Dimensional analysis gives an even farther-reaching indication: there must be something at work which the classical constants m , $e^2/4\pi\epsilon_0$ and c cannot quantify: They can be combined to yield

$$r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.8 \text{ fm},$$

while all other length scales are even smaller than this four-orders-of-magnitude-too-small one. In other words, there is no classical length the atomic size Rutherford was probing. [Estimate the order of magnitude and unit of a new constant of nature that could set the atomic scale $a_0 \approx 0.529 \text{ \AA}$.]



In turn, once a this novel mechanism introduces the new length scale, the stability of the atomic nucleus can/could be a consequence of the same mechanism.

But I want to start with a tangible example which should embody the key idea which represents the fundamental shift in philosophy that quantum mechanics introduces:

Classical physics: a system has properties, and measuring reveals them.

Quantum physics: a system has a state; a measurement is a question; some questions are incompatible, and answering one erases the answer to another.

It remains to specify (mathematically):

- How the state is represented? [vector]
- How is the state measured, in general, manipulated? [expectation values and operators]
- How is its evolution in time predicted? [Action principle $\rightarrow$ Schrödinger equation, &c.]

measurements on two-state systems

Let us begin by putting on Stern and Gerlach's shoes, and send Atoms into an inhomogeneous magnetic field.

[SIM] [stern-gerlach.html](#)

More generally, this asks what is being detected if a beam of particles, presumably all in the same state, is sent through an appropriate version of a Stern-Gerlach apparatus? First, let us run the experiment such that we can trace each particle, silver atom or photon or ..., individually. We find

$$\sigma(\hat{n}) = \pm 1 \quad \text{for every filter orientation } \hat{n}, \quad \text{never } 0, \quad \text{never } \cos \theta.$$

$[\sigma(\hat{n})]$: the reading of the apparatus whose axis points along the unit vector $\hat{n}$

and a spatial splitting of the beam; the classically-expected band is not found.

What happens if the apparatus is rotated? Nothing, at all, for we would still detect only ± 1 for each particle. Now, let us add another Stern-Gerlach apparatus in sequence, with its axis oriented along a different direction. The idea is to allow only one of the two states exiting the first filter to enter the second. This is what the above simulation does when the oven emits particles with a defined "orientation". Note that although we call it orientation, we do not yet know what we are dealing with except that "it" apparently dictates whether our atom ends up in the "up" or "down" path of the Stern-Gerlach apparatus.

You should now find what the two guys found long ago, namely not the

$$\underbrace{\sigma_{\text{cl}} = \hat{n} \cdot \hat{m} \in [-1, 1]}_{\text{expected, continuous}} \quad \text{but the discrete } \sigma = \pm 1 \quad \text{with the special case } \sigma|_{\hat{n}=\hat{m}} = +1.$$

$[\hat{m}]$: the preparation (polarizer) axis, $\hat{n}$: the analysis (analyzer) axis, both unit 3-vectors in ordinary space.]

Discuss the most general arrangement: polarizer oriented in $\hat{m}$ and analyzer oriented in $\hat{n}$ direction. For a single particle, the outcome is ± 1 , for all (!) orientations of oven and filter (do not confuse the behind-the-filter blocking with the actual measurement outcome).

Considering an ensemble of identically prepared particles along $\hat{m}$, we observe an average outcome

$$\langle \sigma \rangle = \hat{n} \cdot \hat{m} = \cos \theta, \quad \theta = \angle(\hat{m}, \hat{n}),$$

$[\langle \cdot \rangle]$: arithmetic mean of single-particle outcomes] which seems to measure the component of the preparation axis $\hat{m}$ along the analysis axis $\hat{n}$.

Let us now cast this in the language of probabilities: “How probable is it to get $\sigma = +1$ or $\sigma = -1$ for given ' $\hat{m}$ and $\hat{n}$ ': [$P_{\pm}$: probability that the analyzer along $\hat{n}$ reads $\sigma = \pm 1$, for a particle prepared along $\hat{m}$. They sum to one because no third reading ever occurs.]

$$P_+ + P_- = 1, \quad P_+ - P_- = \cos \theta \quad \implies \quad P_{\pm} = \frac{1 \pm \cos \theta}{2}, \quad P_+ = \cos^2 \frac{\theta}{2}, \quad P_- = \sin^2 \frac{\theta}{2},$$

$$\theta = 0 : (1, 0); \quad \theta = \frac{\pi}{2} : \left(\frac{1}{2}, \frac{1}{2}\right); \quad \theta = \pi : (0, 1).$$

Let us proceed with a setup designed to shed light on whether or not the measurements erase information encoded in the initial preparation along $\hat{z}$. With the sequence

$$\hat{z} \longrightarrow \hat{x} \longrightarrow \hat{z} \quad \text{we find} \quad \langle \sigma_z \rangle = 0, \quad ,$$

[$\sigma_z \equiv \sigma(\hat{z})$, and likewise σ_x, σ_y : the reading with the apparatus along a coordinate axis. The arrows are successive settings of one and the same apparatus, in time order. Susskind §1.4.] and thus, if we allow only the $+\hat{z}$ channel to pass through $\hat{x}$, the right $\hat{z}$ result $\langle \sigma_z \rangle = 0$ implies equal probabilities for the $+\hat{z}$ and $-\hat{z}$ outcomes.

You can analyze this with

[SIM] [stern-gerlach-three.html](#)

which you (sh/c)ould parametrize such that it reproduces the original 2-filter experiment.

Having seen all this in a context (atoms) where a different outcome was expected, recall that the observed behavior is what is classically accepted when dealing with photons and their polarization. The latter is described by the three sets of orthogonal polarization states: vertical/horizontal, diagonal/anti-diagonal, and left-/right-circular, which correspond to the three ways the atoms leave the Stern-Gerlach apparatus along three mutually orthogonal axes (magnetic fields along $\hat{x}, \hat{y}, \hat{z}$). The following pairs represent the corresponding polarization directions of the electric/magnetic field of the photon:

$$\{\hat{v}, \hat{h}\}, \quad \{\hat{d}, \hat{a}\}, \quad \{\hat{r}, \hat{l}\},$$

[$\hat{v}, \hat{h}, \hat{d}, \hat{a}, \hat{r}, \hat{l}$: the six analyzer settings vertical, horizontal, diagonal, anti-diagonal, right-circular, left-circular.] and if a photon is polarized along any one of these, the probability to find it in any state of the other two pairs is $P = \frac{1}{2}$. If the angle between initial and analyzing polarizer is θ , the probability to find the photon in the analyzing polarizer is given by Malus' law which Stern and Gerlach found for their atomic beams, too:

$$P_+ = \cos^2 \theta \quad \text{and} \quad \langle \sigma \rangle = \cos 2\theta.$$

These observations are now to be explained. They suffice to guess/postulate the formal structure of QM. First, let us get to the stage of knowledge from where we can make such a conjecture: what are complex numbers, and vector spaces (bra, kets, inner products, operators, and bases)?

complex numbers

[Susskind §1.8.]

$$z = x + iy, \quad x, y \in \mathbb{R}, \quad i^2 = -1, \quad z = r e^{i\theta} = r (\cos \theta + i \sin \theta).$$

[$r = \sqrt{z^* z} \geq 0$ is called modulus, and θ argument. Two real numbers represent one complex number.] Addition is easier in components, multiplication in polar form:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \quad (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

Every z has a unique conjugate/dual z^* , with z^*z real and non-negative:

$$z = x + iy = r e^{i\theta} \iff z^* = x - iy = r e^{-i\theta} , \quad z^*z = r^2 \geq 0 .$$

We denote those numbers with $r = 1$,

$$z^*z = 1 , \quad z = e^{i\theta} = \cos \theta + i \sin \theta ,$$

phase factors or simply phases.

vector spaces

The measurement outcomes discussed above suggest the following, profound switch:

$$\text{classical system} \leftrightarrow \text{set} \quad \Leftrightarrow \quad \text{quantum system} \leftrightarrow \text{vector}$$

Let us denote these state vectors “kets”, and use the notation:

$$|\text{set of numbers which characterize the state}\rangle \stackrel{\text{e.g.}}{=} |\text{position, mass, (hair) color, (eye) color, \&c.}\rangle .$$

If vectors, those objects must satisfy the axioms of a vector space:

- (i) $|A\rangle + |B\rangle = |C\rangle$
- (ii) $|A\rangle + |B\rangle = |B\rangle + |A\rangle$
- (iii) $\{ |A\rangle + |B\rangle \} + |C\rangle = |A\rangle + \{ |B\rangle + |C\rangle \}$
- (iv) $\exists! 0 : |A\rangle + 0 = |A\rangle$
- (v) $\exists! -|A\rangle : |A\rangle + (-|A\rangle) = 0$
- (vi) $|zA\rangle = z|A\rangle = |B\rangle$
- (vii) $z\{ |A\rangle + |B\rangle \} = z|A\rangle + z|B\rangle , \quad \{ z + w \} |A\rangle = z|A\rangle + w|A\rangle$

Two-dimensional state vectors can be represented concretely as columns of complex numbers:

$$|A\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} , \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \end{pmatrix} , \quad z \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} z\alpha_1 \\ z\alpha_2 \end{pmatrix} .$$

$[\alpha_i, \beta_i \in \mathbb{C}$: the components of $|A\rangle$ and $|B\rangle$ with respect to a basis that has been fixed beforehand while the abstract ket refers to the state itself without reference to any basis.]

To every “ket” belongs a “bra” with an *antilinear* correspondence:

$$|A\rangle \longleftrightarrow \langle A| , \quad |A\rangle + |B\rangle \longleftrightarrow \langle A| + \langle B| , \quad z|A\rangle \longleftrightarrow \langle A| z^* ,$$

$$|A\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \iff \langle A| = (\alpha_1^* \quad \cdots \quad \alpha_N^*) .$$

The inner product is always bra times ket, and returns a complex number:

$$\begin{aligned} \langle C| \{ |A\rangle + |B\rangle \} &= \langle C|A\rangle + \langle C|B\rangle , & \langle B|A\rangle &= \langle A|B\rangle^* , \\ \langle B|A\rangle &= \sum_i \beta_i^* \alpha_i , & \langle A|A\rangle &= 1 \text{ (normalized)} , & \langle B|A\rangle &= 0 \text{ (orthogonal)} . \end{aligned}$$

The dimension of the space is the largest number of mutually orthogonal vectors in it; an orthonormal basis $|i\rangle$, $i = 1 \dots N$, then resolves any ket, the components being its inner products with the basis vectors: $[\delta_{ij}$: the Kronecker symbol, 1 for $i = j$ and 0 otherwise — here it is precisely the statement that the basis is orthonormal. Susskind §1.9.5, Eqs. 1.3–1.5.]

$$\langle j|i\rangle = \delta_{ij} , \quad |A\rangle = \sum_i \alpha_i |i\rangle , \quad \alpha_j = \langle j|A\rangle , \quad |A\rangle = \sum_i |i\rangle \langle i|A\rangle .$$

[$\sqsupset_I$] Mapping physics onto mathematics

Now let us map physics onto math: the state ket-vector identification (Dirac and row/column notation). Specifically, spin states: all there is to know, there is to measure, are the components along a given direction in space, and since any such direction in 3 dimensions is specified by two numbers, the state space is two-dimensional. Express the measurement outcomes for the xyz directions using Dirac notation $|\uparrow_x\rangle, |\downarrow_x\rangle, |\uparrow_y\rangle, |\downarrow_y\rangle, |\uparrow_z\rangle, |\downarrow_z\rangle$ and phrase that in terms of three pairs of mutually orthogonal vectors whose projections on one another are identical. As a consequence: the vector space is complex and two-dimensional.

[R_I] Reading list

- [1] O. Stern, “A Way towards the Experimental Examination of Spatial Quantisation in a Magnetic Field”, trans. by E. E. B. Campbell and B. Friedrich, *Zeitschrift für Physik D: Atoms, Molecules and Clusters* **10**, 114–116 (1988) doi:10.1007/BF01384842; trans. of “Ein Weg zur experimentellen Prüfung der Richtungsquantelung im Magnetfeld”, *Zeitschrift für Physik* **7**, 249–253 (1921) doi:10.1007/BF01332793.
- [2] W. Gerlach and O. Stern, *The Stern-Gerlach Experiment*, trans. by M. Bauer, (2023) arXiv:2301.11343 [physics.hist-ph]; trans. of “Der experimentelle Nachweis der Richtungsquantelung im Magnetfeld”, *Zeitschrift für Physik* **9**, 349–352 (1922) doi:10.1007/BF01326983.
- [3] G. E. Uhlenbeck and S. Goudsmit, “Spinning Electrons and the Structure of Spectra”, *Nature* **117**, 264–265 (1926) doi:10.1038/117264a0.
- [4] R. P. Feynman, F. L. Vernon, and R. W. Hellwarth, “Geometrical Representation of the Schrödinger Equation for Solving Maser Problems”, *Journal of Applied Physics* **28**, 49–52 (1957) doi:10.1063/1.1722572.
- [5] L. Susskind and A. Friedman, *Quantum Mechanics: The Theoretical Minimum* (Basic Books, 2014), chapters 1–3 [pedagogical].
- [6] G. Baym, *Lectures on Quantum Mechanics* (W. A. Benjamin, 1969), chapter 1 up to “Angular Momentum” [pedagogical].

[P_I] Problem list

Level 1

- P 1. Three Stern-Gerlach filters (SG_i).** A beam leaves the oven with no preparation axis. It passes SG_1 along $\hat{z}$, of which only the $\sigma = +1$ channel continues; then SG_2 along $\hat{n}$ at an angle θ to $\hat{z}$, again keeping only $\sigma = +1$; then SG_3 along $\hat{z}$.
- (a) Applying $P_{\pm}(\theta) = \cos^2 \frac{\theta}{2}, \sin^2 \frac{\theta}{2}$ once at each analyser, give the fraction of the oven output reaching each of the two SG_3 channels.
 - (b) Evaluate at $\theta = 0, \pi/2, \pi$. At which θ is the $\sigma_z = -1$ count largest, and what is it?
 - (c) Remove SG_2 and give the same two numbers.
 - (d) Suppose every atom carried a fixed label λ and each analyser passed or stopped it according to λ alone. Transmission would then be intersection of sets. Show $T_{123} = T_1 \cap T_2 \cap T_3 \subseteq T_1 \cap T_3$, so inserting SG_2 cannot raise any count. Compare with (b) and (c).

[SIM] [stern-gerlach-three.html](#) runs the chain; check your (a) against the counter before you trust either.

P 2. Measurements on single photons and on ensembles. A polarizer and an analyser have their transmission axes at θ_{lab} .

- (a) For a beam, $I/I_0 = \cos^2 \theta_{\text{lab}}$; for one photon, $P = \cos^2 \theta_{\text{lab}}$. The same function is being used twice. State precisely what each of the two equations asserts, and what is being counted in each.
- (b) N photons are sent one at a time. The transmitted number K is binomial with $p = \cos^2 \theta_{\text{lab}}$. Give $\langle K \rangle$ and $\text{Var } K$.
- (c) Show $\sqrt{\text{Var } K} / \langle K \rangle = \sqrt{(1-p)/(Np)}$. At $\theta_{\text{lab}} = 45^\circ$, how large must N be before the measured transmission matches Malus to one percent?
- (d) Attenuating a laser until on average one photon per second arrives does *not* make it a one-photon source. Say why, and name a measurement that separates the two.

Level 2

P 3. The three bases. In $\mathbb{C}^2$ fix an orthonormal pair $|z+\rangle, |z-\rangle$ and define

$$|x\pm\rangle = \frac{1}{\sqrt{2}}(|z+\rangle \pm |z-\rangle), \quad |y\pm\rangle = \frac{1}{\sqrt{2}}(|z+\rangle \pm i|z-\rangle).$$

- (a) Verify that each of the three pairs is orthonormal.
- (b) Compute $|\langle a|b\rangle|^2$ for every a taken from one pair and b from another. Show that all twelve numbers equal $\frac{1}{2}$.
- (c) The photon bases $\{|H\rangle, |V\rangle\}$, $\{|D\rangle, |A\rangle\}$, $\{|R\rangle, |L\rangle\}$ obey the same relations. Write $|D\rangle, |A\rangle, |R\rangle, |L\rangle$ in terms of $|H\rangle, |V\rangle$ and repeat (b) in that language.
- (d) On the sphere a basis is an axis, and $|\langle a|b\rangle|^2 = \cos^2 \frac{\theta}{2}$ with θ the angle between axes. Show $|\langle a|b\rangle|^2 = \frac{1}{2}$ holds exactly when the axes are perpendicular, and conclude that $\mathbb{C}^2$ admits at most three such bases.

P 4. discreteness from an anti-commutator. Let $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the Pauli matrices and $\hat{n}$ a real unit vector with polar angles (θ, ϕ) .

- (a) Write $\sigma_n = \hat{n} \cdot \vec{\sigma}$ as a 2×2 matrix.
- (b) Using $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\mathbf{1}$, show $\sigma_n^2 = \mathbf{1}$ for every $\hat{n}$. Deduce that the only possible readings are ± 1 — without solving an eigenvalue problem.
- (c) Now solve for the eigenvectors and show, up to an overall phase,

$$|\hat{n}+\rangle = \cos \frac{\theta}{2} |z+\rangle + e^{i\phi} \sin \frac{\theta}{2} |z-\rangle, \quad |\hat{n}-\rangle = -\sin \frac{\theta}{2} |z+\rangle + e^{i\phi} \cos \frac{\theta}{2} |z-\rangle.$$

- (d) Compute $|\langle \hat{n}+|z+\rangle|^2$ and recover $\cos^2 \frac{\theta}{2}$.
- (e) In the state $|\hat{n}+\rangle$, compute $\langle \sigma_n \rangle$ and show it equals $\hat{n} \cdot \hat{m}$.

[Susskind Lecture 3, Exercises 3.3 and 3.4, worked in his §3.7.]

Level 3

P 5. The Bloch sphere.

- (a) Show that $|\psi\rangle$ and $e^{i\gamma}|\psi\rangle$ give identical probabilities for every measurement, so the physical state is the ray and not the vector.

- (b) A normalised vector in $\mathbb{C}^2$ carries four real numbers; normalisation and the global phase remove two. Show that $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$ with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ covers every ray exactly once. Mind the poles.
- (c) Show that $\langle \vec{\sigma} \rangle = \hat{n}$ in the state $|\hat{n}+\rangle$: the Bloch vector is the expectation value of $\vec{\sigma}$.
- (d) Show that orthogonal states are antipodal, and mark the six states of P 3 on a sphere.
- (e) A polarization ellipse returns to itself after a laboratory rotation of π , while the state vector returns only after 2π and collects a minus sign at π . Explain how both are true at once, and connect it to the 2π rotation of a spin- $\frac{1}{2}$.
- (f) Why is the set of rays in $\mathbb{C}^2$ called $\mathbb{CP}^1$, and why is $\mathbb{CP}^1 \cong S^2$? Build the stereographic map explicitly.

P 6. Labels as an alternative model. A concrete classical model: each atom leaves the oven carrying a unit vector $\vec{\lambda}$, isotropically distributed; an analyser along $\hat{n}$ reports $\sigma = \text{sign}(\vec{\lambda} \cdot \hat{n})$; a filter keeps the atoms it reports +1 for and discards the rest, doing nothing else to $\vec{\lambda}$.

- (a) For the unprepared oven beam, show $P_{\pm} = \frac{1}{2}$ for every $\hat{n}$ and $\langle \sigma \rangle = 0$. The model survives this test — which is why the first experiment alone decides nothing.
- (b) Prepare by keeping the + channel along $\hat{m}$, so $\vec{\lambda} \cdot \hat{m} > 0$, and then measure along $\hat{n}$ at angle θ . Show that the two hemispheres overlap in a lune and hence

$$P_+ = 1 - \frac{\theta}{\pi}, \quad \langle \sigma \rangle_{\text{model}} = 1 - \frac{2\theta}{\pi}.$$

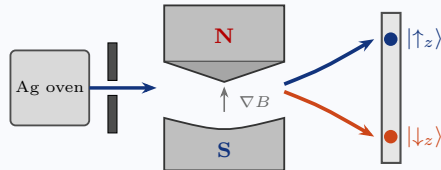
- (c) Compare with the measured $\langle \sigma \rangle = \cos \theta$. At which θ do the two agree exactly? Evaluate both at $\theta = \pi/3$, and find the θ at which they differ most.
- (d) Apply the model to the chain of P 1 and show that it gives *zero* in the $\sigma_z = -1$ channel for every θ . Identify the single assumption responsible, and state what the quantum account does in its place.

[T_I] The I-never-read-textbooks issues

- Isn't it surprising that we have already the means to predict the expected/average outcome of a system prepared in a certain state without having specified what the measurement does to the system? There is a measurement operator, yes, but that tells us only what kind of measurements are possible, and how likely those are (eigensystem). However, what state the system is left in post-measurement has not been discussed. But you should have given enough information to make a conjecture which state the system will be in if a particular measurement outcome is obtained.

Quantum Mechanics I – Week I: 2-state systems and the Bloch Sphere

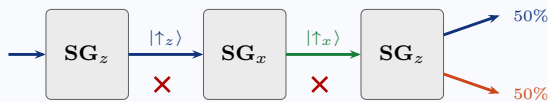
2- state atoms: Spin-(1/2)



Measurement outcomes

- classical: $\vec{\mu} \cdot \hat{n} \in [-1, 1]$, continuous
- quantum: $\sigma(\hat{n}) = \pm 1$, discrete and $\hat{n}$ independent

Sequential filtering: incompatibility



State erasure: $\hat{x}$ wipes out the prior $\hat{z}$

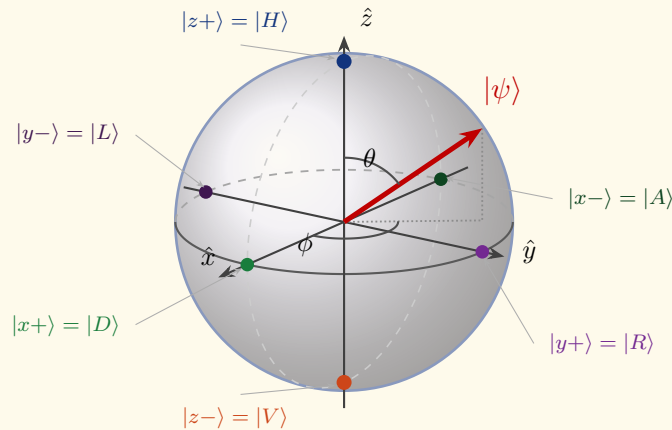
Without SG_x no counts in the lower channel. Adding SG_x counts as many clicks in the lower channel as in the upper channel.

$$T_{zz}^{\downarrow/\uparrow} = \frac{1}{4} \neq 0 \Rightarrow \langle \sigma_z \rangle = 0 \quad \text{vs.} \quad T_{zz}^{\downarrow} = 0$$

$$p_{\pm}(\theta) = \cos^2 \frac{\theta}{2}, \quad \sin^2 \frac{\theta}{2} \quad \langle \sigma \rangle = \hat{n} \cdot \hat{m} = \cos \theta$$

$\theta = \angle(\hat{m}, \hat{n})$: angle between preparing and analyzing axes.

2-state systems: Bloch sphere



$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$

$$\{|z\pm\rangle\} = \{|H\rangle, |V\rangle\} \quad \{|x\pm\rangle\} = \{|D\rangle, |A\rangle\} \quad \{|y\pm\rangle\} = \{|R\rangle, |L\rangle\}$$

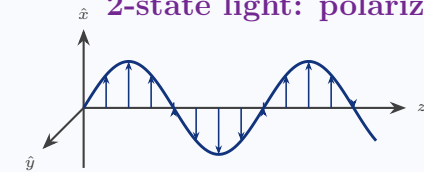
the two states are	angle apart	$ \langle a b\rangle ^2$	SG reads
the same state	0°	1	always +
partners in one basis	180°	0	never
from two bases	90°	1/2	50/50

$$|\langle \hat{n} s | \hat{m} s' \rangle|^2 = \frac{1}{2} (1 + s s' \hat{n} \cdot \hat{m})$$

$$s, s' = \pm 1, \quad \hat{n} \cdot \hat{m} = \cos \theta_{\text{Bloch}} = \cos 2\theta_{\text{lab}} \Rightarrow \cos^2 \frac{\theta}{2}, \quad \sin^2 \frac{\theta}{2}$$

Orthogonal states are antipodal; a basis is an axis; axes are perpendicular.

2-state light: polarization

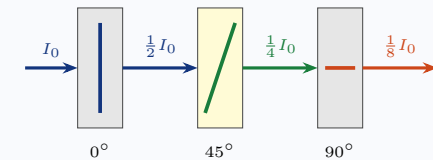


$$\vec{E}(z, t) = \text{Re}[\vec{E}_0 e^{i(kz - \omega t)}]$$

$$\vec{E}_0 = E_0 (\cos \alpha \hat{x} + e^{i\delta} \sin \alpha \hat{y})$$

basis	α	δ	$\vec{E}_0 / E_0$
$ H\rangle, V\rangle$	0, $\frac{\pi}{2}$	–	$\hat{x}; \hat{y}$
$ D\rangle, A\rangle$	$\frac{\pi}{4}$	0, π	$\frac{1}{\sqrt{2}}(\hat{x} \pm \hat{y})$
$ R\rangle, L\rangle$	$\frac{\pi}{4}$	$\pm \frac{\pi}{2}$	$\frac{1}{\sqrt{2}}(\hat{x} \pm i \hat{y})$

The three-filter experiment



Single photon clicks

- the detector registers indivisible clicks $\hbar\omega$
- Malus' law governs the individual probability:

$$p(\text{click}) = |\langle \text{filter} | \psi \rangle|^2 = \cos^2 \theta_{\text{lab}}$$

- additional filter raises a count:

$$T_{0^\circ \rightarrow 45^\circ \rightarrow 90^\circ} = \frac{1}{8} \neq 0 \quad \text{vs.} \quad T_{0^\circ \rightarrow 90^\circ} = 0$$

1. HILBERT SPACE $\mathbb{C}^2$

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle, \quad |c_0|^2 + |c_1|^2 = 1$$

$$\text{Dual bra: } \langle \psi| = c_0^* \langle 0| + c_1^* \langle 1|$$

$$\text{Inner product: } \langle \phi | \psi \rangle = \phi_0^* \psi_0 + \phi_1^* \psi_1 \stackrel{!}{=} \langle \psi | \phi \rangle^*$$

2. OBSERVABLES AS OPERATORS

$$A = A^\dagger = \sum_i a_i |i\rangle \langle i|, \quad a_i \in \mathbb{R}$$

$$\text{Eigenvalues} = \text{outcomes } (a_i = \pm 1)$$

$$\text{Spin: } A = \vec{\sigma} \cdot \hat{n}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. BORN RULE & PROJECTORS

$$p_i = |\langle i | \psi \rangle|^2 = \langle \psi | i \rangle \langle i | \psi \rangle = \langle \psi | P_i | \psi \rangle$$

$$\text{Projectors: } P_i = |i\rangle \langle i|, \quad P_i^2 = P_i = P_i^\dagger$$

$$\text{Identity: } \sum_i P_i = \mathbf{1}$$

$$\text{Expectation: } \langle A \rangle = \langle \psi | A | \psi \rangle = \sum_i a_i p_i$$

4. MEASUREMENT COLLAPSE

$$|\psi\rangle \xrightarrow{\text{outcome } i} \frac{P_i |\psi\rangle}{\|P_i |\psi\rangle\|} = |i\rangle$$

Post-measurement state is eigenstate $|i\rangle$

Measurement prepares the state

[N_{II}] Time development of a state vector

After having defined the mathematical representation of a quantum system whose state can assume any element of a state space which is a complex vector space, and after having defined how measurements reveal properties of the system in a specific state (nothing has been said yet about the state of the system post-measurement!), a complete theory must also describe the temporal evolution of the system.

I will present this evolution in the abstract on a general state vector $|\Psi(t)\rangle$, and for the specific case when this state vector does describe the spatial location of the system. Thus far, we only dealt with a 2-state system, and you need to generalize this notion such that a measurement can not only return two outcomes corresponding to up/down, for instance, but any location along a line or in space, in general.

[Susskind & Friedman, Lecture 4, §§4.1–4.6]

It is a reasonable assumption that the evolution of a state from $t = 0$ to $t > 0$ is represented by a linear operator:

$$|\Psi(t)\rangle = U(t) |\Psi(0)\rangle \quad (4.1)$$

The first constraint we want to impose on $U(t)$ makes it preserve the distinctness of states:

$$\begin{aligned} \langle \Psi(0) | \Phi(0) \rangle &= 0 \quad , \\ \langle \Psi(t) | \Phi(t) \rangle &= 0 \quad . \end{aligned} \quad (4.2)$$

From this requirement, we can infer:

$$\langle \Psi(t) | = \langle \Psi(0) | U^\dagger(t) \quad (4.3)$$

$$\langle \Psi(0) | U^\dagger(t) U(t) | \Phi(0) \rangle = 0 \quad . \quad (4.4)$$

If we take the states to be orthonormal, e.g., as elements of an orthonormal basis, we get:

$$\begin{aligned} \langle i | j \rangle &= \delta_{ij} \\ \langle i | U^\dagger(t) U(t) | j \rangle &= 0 \quad (i \neq j) \\ \langle i | U^\dagger(t) U(t) | j \rangle &= \delta_{ij} \end{aligned}$$

and it follows that $U(t)$ must be a so-called unitary operator:

$$U^\dagger U = I$$

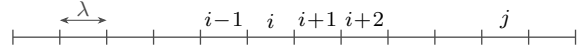
Let us now consider infinitesimally small time increments for which

$$U^\dagger(\epsilon) U(\epsilon) = I \quad (4.5)$$

holds, too, but we can also expand

[Baym, Ch. 3, pp. 46–53]

Instead of $|\Psi(t)\rangle$ being a linear combination of vertical/horizontal, or diagonal/anti-diagonal, or right/left states at each time, let us imagine a system which can occupy any bin of width λ along a one-dimensional line.



As we are describing a particle, it must be somewhere which implies that the sum over the squared overlap of the particle's state with the state $|i\rangle$ corresponding to bin i , $\psi_i(t) \equiv \langle i | \Psi(t) \rangle$, must be unity:

$$\sum_i |\psi_i(t)|^2 = 1 \quad (3-1)$$

Instead of the 2-component column vector, we are now dealing with the ∞ -component column

$$|\Psi\rangle = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \\ \psi_{i-1}(t) \\ \psi_i(t) \\ \psi_{i+1}(t) \\ \vdots \end{pmatrix} \quad (3-2)$$

It is intuitive to think of time evolution of the state through leakage of overlap (probability amplitude) from bin to bin, and to assume (why?) that the amount of this leakage from bin i to bin $i + 1$ in Δt is

$$-i \Delta t w_{i+1,i} \psi_i(t) \quad (3-3)$$

with $(-i)$ being factored out of the coefficients $w_{i+1,i}$, $w_{i,i-1}$, $w_{i,i+1}$, and w_{ii} to make them furnish a hermitian matrix. It is now straightforward to represent the amplitude after a time increment Δt by

$$U(\epsilon) = I - i\epsilon H \quad (4.6)$$

$$U^\dagger(\epsilon) = I + i\epsilon H^\dagger \quad (4.7)$$

and find that unitarity of $U(\epsilon)$ requires H to be Hermitian:

$$\begin{aligned} (I + i\epsilon H^\dagger)(I - i\epsilon H) &= I \\ H^\dagger - H &= 0 \\ H^\dagger &= H \end{aligned} \quad (4.8)$$

The infinitesimal evolution of the state vector is expressed in terms of H :

$$\begin{aligned} |\Psi(\epsilon)\rangle &= |\Psi(0)\rangle - i\epsilon H |\Psi(0)\rangle \\ \frac{|\Psi(\epsilon)\rangle - |\Psi(0)\rangle}{\epsilon} &= -iH |\Psi(0)\rangle \end{aligned}$$

and in the limit $\epsilon \rightarrow 0$, we obtain the time-dependent Schrödinger equation:

$$\frac{\partial |\Psi\rangle}{\partial t} = -iH |\Psi\rangle \quad (4.9)$$

This equation implies that the operator H has the dimensions of inverse time, but in order to have units of energy in conventional units, a constant with dimensions $\text{mass} \cdot \text{length}^2 / \text{time}$ must be introduced. This constant is Planck's constant $\hbar$, whose numerical value is chosen to yield numbers of order unity for typical quantum mechanical systems:

$$\begin{aligned} h &\approx 6.6 \times 10^{-34} \text{ kg m}^2/\text{s} \\ \hbar &= \frac{h}{2\pi} = 1.054571726 \dots \times 10^{-34} \text{ kg m}^2/\text{s} \\ \left[\frac{\partial}{\partial t} \right] &= \text{s}^{-1}, \quad [H] = \text{kg m}^2/\text{s}^2, \quad [\hbar] = \text{kg m}^2/\text{s} \\ \hbar \frac{\partial |\Psi\rangle}{\partial t} &= -iH |\Psi\rangle \end{aligned} \quad (4.10)$$

$$\begin{aligned} \psi_i(t + \Delta t) &= \psi_i(t) - i \Delta t w_{i,i-1} \psi_{i-1}(t) \\ &\quad - i \Delta t w_{i,i+1} \psi_{i+1}(t) \\ &\quad - i \Delta t \underbrace{(-w_{i,i+1} - w_{i,i-1} + v_i)}_{=: w_{ii}} \psi_i(t) \end{aligned} \quad (3-4)$$

with the first term representing what has been there at t , the second and third terms influx from left and right. The term $\propto \psi_i(t)$ represents outflux from bin i and repulsion/attraction due to the “potential” of site i , v_i .

First, let us examine constraints on the bin-to-bin transition rates as they follow from

$$\sum_i |\psi_i(t + \Delta t)|^2 = \sum_i |\psi_i(t)|^2 \quad (3-5)$$

$$\begin{aligned} &\sum_i |\psi_i(t)|^2 (w_{ii} - w_{ii}^*) \\ &+ \sum_i \psi_i^*(t) \psi_{i+1}(t) (w_{i,i+1} - w_{i+1,i}^*) \\ &+ \sum_i \psi_i^*(t) \psi_{i-1}(t) (w_{i,i-1} - w_{i-1,i}^*) = 0 \end{aligned} \quad (3-6)$$

which yields:

$$w_{ii} = w_{ii}^*, \quad w_{i,i+1} = w_{i+1,i}^* \quad (3-7)$$

In the more practical matrix notation, we can write this as

$$|\psi(t + \Delta t)\rangle = (1 - i \Delta t W) |\psi(t)\rangle \quad (3-9)$$

with

$$W = \begin{pmatrix} w_{11} & w_{12} & & & \\ w_{12}^* & w_{22} & w_{23} & & \\ & w_{23}^* & w_{33} & w_{34} & \\ & & w_{34}^* & w_{44} & w_{45} \\ & & & \ddots & \ddots \end{pmatrix} \quad (3-8)$$

Let us now take the continuous-time limit in Eq. (3-4):

$$i \frac{\partial \psi_i(t)}{\partial t} = w_{i,i-1} \psi_{i-1}(t) + w_{i,i+1} \psi_{i+1}(t) + w_{ii} \psi_i(t)$$

and resolve w_{ii} such that one gets a very conspicuous form

$$\begin{aligned} i \frac{\partial \psi_i(t)}{\partial t} &= w_{i,i-1} [\psi_{i-1}(t) - \psi_i(t)] \\ &\quad + w_{i,i+1} [\psi_{i+1}(t) - \psi_i(t)] + v_i \psi_i(t) \end{aligned} \quad (3-12)$$

which reads in matrix notation

$$i \frac{\partial}{\partial t} |\Psi(t)\rangle = W |\Psi(t)\rangle$$

So in both columns, we arrive at the same structure for a differential equation. On the right, we can make the analogy with the heat equation, or any diffusion-like process characterized by the diffusion coefficient κ :

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r}, t)$$

Let us elaborate a little on this analogy as I find it easier to think of actual stuff flowing instead of “probability amplitudes”. So we are allowed to identify $\psi_i(t)$ with some actual quantity, e.g., air, that flows. In a stationary situation without any inflow or outflow, we have can solve for $\psi_i(t)$ as

$$\psi_i(t) = e^{-iv_it/\hbar} \psi_i(0) \quad (3-13)$$

As air flows continuously, we would like to take the limit $\lambda \rightarrow 0$ during which we need to be careful with the scaling of the discrete (unit-less because its square yields a dim-less probability) vs. continuous (with units of $\text{length}^{-1/2}$ as its square integrated over the line now yields the probability) representations:

$$\begin{aligned} \psi_i &\rightarrow \sqrt{\lambda} \psi(x_i) \\ [\psi_i] &= 1, \quad [\psi(x)] = (\text{length})^{-1/2} \end{aligned}$$

Analogously to taking $\Delta t \rightarrow 0$, we replace the indices with arguments, and take the continuous limit $\lambda \rightarrow 0$:

$$i \frac{\partial}{\partial t} \psi(x_i, t) = w_{i,i-1} [\psi(x_{i-1}, t) - \psi(x_i, t)] + w_{i,i+1} [\psi(x_{i+1}, t) - \psi(x_i, t)] + v(x_i, t) \psi(x_i, t)$$

with $x_{i+1} = x_i + \lambda$ and the ensuing expansion

$$\psi(x_{i\pm 1}, t) = \psi(x_i, t) \pm \lambda \frac{\partial \psi(x_i, t)}{\partial x_i} + \frac{\lambda^2}{2} \frac{\partial^2 \psi(x_i, t)}{\partial x_i^2} + \dots \quad (3-15)$$

we obtain

$$i \frac{\partial}{\partial t} \psi(x_i, t) = \lambda [w_{i,i+1} - w_{i,i-1}] \frac{\partial \psi(x_i, t)}{\partial x_i} + \frac{\lambda^2}{2} [w_{i,i+1} + w_{i,i-1}] \frac{\partial^2 \psi(x_i, t)}{\partial x_i^2} + \frac{v(x_i, t)}{\hbar} \psi(x_i, t)$$

which we can rewrite further by introducing strength functions characterizing the likelihood to hop to the left or to the right into any position x along the line.

$$w_{i,i+1} = w_L(x_i), \quad w_{i,i-1} = w_R(x_i) \quad (3-17)$$

$$w_R(x) = w_L(x)^* \quad (3-18)$$

Note that this does not mean that the hopping rates to the left and right are equal. The complex conjugation resembles an inversion of direction (see Eq. (3-7)).

$$i \frac{\partial \psi(x, t)}{\partial t} = \frac{\lambda^2}{2} [w_L(x) + w_R(x)] \frac{\partial^2 \psi(x, t)}{\partial x^2} + \lambda [w_L(x) - w_R(x)] \frac{\partial \psi(x, t)}{\partial x} + \frac{1}{\hbar} v(x, t) \psi(x, t) \quad (3-19)$$

However, unequal hopping corresponds to a preference for hopping in one direction over the other due to the property of the line. If we believe that this is the frictional force often disregarded in first approximation, we would set

$$w_R(x) = w_L(x) = w \quad .$$

Let us now be concrete, and try to relate this hopping rate to characteristics of the system we are describing, i.e., a single point particle in one spatial dimension subjected to nothing but the ominous tendency to hop, and which were are more familiar with.

To this end, let us find a solution to Eq. (3-19) in the case of a free particle, i.e., when $v(x, t) = 0$, and

$$i \frac{\partial \psi(x, t)}{\partial t} = \frac{\lambda^2}{2} w \frac{\partial^2 \psi(x, t)}{\partial x^2}$$

For this equation, the freely oscillating plane wave is given by

$$\psi(x, t) = e^{i(kx - Et)}$$

and represents a solution if

$$E = \frac{\lambda^2}{2} w k^2$$

Now for the photon we know that $p = \hbar k$ and that in some limit which takes us from quantum to classical, we should obtain $E = \frac{p^2}{2m}$. Hence we identify $\frac{\lambda^2}{2} w = \frac{\hbar^2}{2m}$ and obtain the equation of motion for a quantum particle with mass m in one spatial dimension:

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + v(x, t) \psi(x, t) \quad .$$

Expectation values and their time evolution

We learned about the probability that a single measurement yields a particular eigenvalue. Now if we perform multiple measurements on identically prepared systems, we *expect* an outcome:

$$\langle L \rangle = \sum_i \lambda_i P(\lambda_i) \quad .$$

The state in which the system is prepared can be expanded in terms of the eigenstates of L :

$$|A\rangle = \sum_i \alpha_i |\lambda_i\rangle \quad ,$$

and we find nothing new by applying the operator L to the state $|A\rangle$:

$$L |A\rangle = \sum_i \alpha_i L |\lambda_i\rangle = \sum_i \alpha_i \lambda_i |\lambda_i\rangle \quad .$$

In the above notation for the expectation value of L , the state in which this value is calculated is not explicit, but we must remember that it depends on that particular state:

$$\langle L \rangle = \langle A | L | A \rangle = \sum_i (\alpha_i^* \alpha_i) \lambda_i \quad .$$

Let us now consider how this quantity evolves with time:

$$\frac{d}{dt} \langle \Psi(t) | L | \Psi(t) \rangle = \langle \dot{\Psi}(t) | L | \Psi(t) \rangle + \langle \Psi(t) | L | \dot{\Psi}(t) \rangle \quad ;$$

Using the time-dependent Schrödinger equation, we obtain

$$\frac{d}{dt} \langle \Psi(t) | L | \Psi(t) \rangle = \frac{i}{\hbar} \langle \Psi(t) | H L | \Psi(t) \rangle - \frac{i}{\hbar} \langle \Psi(t) | L H | \Psi(t) \rangle \quad ,$$

which can be rewritten by introducing the so-called commutator of H and L :

$$\frac{d}{dt} \langle \Psi(t) | L | \Psi(t) \rangle = \frac{i}{\hbar} \langle \Psi(t) | (H L - L H) | \Psi(t) \rangle \quad , \text{ with}$$

$$L M - M L = [L, M] \quad . \quad \quad \quad (\text{Commutator})$$

In order to interpret

$$\frac{d}{dt} \langle L \rangle = \frac{i}{\hbar} \langle [H, L] \rangle \quad ,$$

we remember that H is also a Hermitian operator and represents an observable, specifically the total energy of the system, as we have seen above in case of the free particle. Now we found that if the order of acting with H and L does not matter, i.e., if $[H, L] = 0$, the expected value of L remains constant in time.

Spin precision

Let us now apply this general formalism for a particle with spin- $\frac{1}{2}$, charge e , and mass m_e as it evolves in the presence of a magnetic field. For this specific problem, it was found—although you can also derive this form from more general principles, it is mostly by observation and guesswork that one finds the appropriate structure for the Hamiltonian—that the Hamiltonian takes the form

$$H = - \left(\frac{e}{m_e c} \right) \boldsymbol{\sigma} \cdot \mathbf{B} \quad ,$$

and for the specific case where the magnetic field is along the z -axis:

$$H = - \left(\frac{e B}{m_e c} \right) \sigma_z \equiv \omega \sigma_z \quad .$$

Eigenstates are thus the familiar spin-up and spin-down states along the z -axis, and the eigenvalues are just multiplied by $-\frac{eB}{m_e c}$:

$$E_{\pm} = \mp \frac{e \hbar B}{2 m_e c}, \quad \text{for } H |z_{\pm}\rangle = E_{\pm} |z_{\pm}\rangle \quad .$$

In order to determine a quantity of undeniable practical interest, namely how the expectation value of the spin measured along the x -, y -, and z -axes evolves in time, we can proceed on multiple routes, and the one I chose is not the fastest. However, it exploits the (possibly surprising) fact of the time evolution operator $U(t, 0)$ obeying the same form of the Schrödinger equation as the state vectors (why?). Hence, we find its explicit form as

$$U(t, 0) = \exp\left(\frac{-i\omega\sigma_z t}{\hbar}\right) \quad .$$

In the following, we will abbreviate $|z\pm\rangle \equiv |\pm\rangle$, and consider the expansion of the at $t_0 = 0$ prepared state

$$|A, t = 0\rangle = c_+ |+\rangle + c_- |-\rangle \quad .$$

The to $t > 0$ evolved state is given by

$$U(t, 0) |A, 0\rangle = |A, t\rangle = c_+ \exp\left(\frac{-i\omega t}{2}\right) |+\rangle + c_- \exp\left(\frac{+i\omega t}{2}\right) |-\rangle \quad .$$

It is intuitive to identify a 3-dimensional vector with components given by the expectation values of the spin along the x -, y -, and z -axes, which tells us at any time what can be expected of measurements of an ensemble of identically prepared systems propagating through the magnetic field for a time t . For this quantity, we must compute the expectation values of the spin components along each axis, and we start with x :

$$\begin{aligned} |\langle x\pm | A, t \rangle|^2 &= \left| \left[\left(\frac{1}{\sqrt{2}} \right) \langle + | \pm \left(\frac{1}{\sqrt{2}} \right) \langle - | \right] \cdot \left[\left(\frac{1}{\sqrt{2}} \right) e^{-i\omega t/2} |+\rangle + \left(\frac{1}{\sqrt{2}} \right) e^{+i\omega t/2} |-\rangle \right] \right|^2 \\ &= \left| \frac{1}{2} e^{-i\omega t/2} \pm \frac{1}{2} e^{+i\omega t/2} \right|^2 = \begin{cases} \cos^2 \frac{\omega t}{2}, & \text{for } |x+\rangle \\ \sin^2 \frac{\omega t}{2}, & \text{for } |x-\rangle \end{cases} \quad . \end{aligned}$$

We have used the expansion of the eigenstates of σ_x in terms of the eigenstates of σ_z as deduced empirically in the first lecture as

$$|x\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle) \quad .$$

With this result, we find

$$\langle \sigma_x \rangle = \left(\frac{\hbar}{2} \right) \cos^2 \left(\frac{\omega t}{2} \right) + \left(\frac{-\hbar}{2} \right) \sin^2 \left(\frac{\omega t}{2} \right) = \left(\frac{\hbar}{2} \right) \cos \omega t \quad ,$$

and similarly

$$\langle \sigma_y \rangle = \left(\frac{\hbar}{2} \right) \sin \omega t, \quad \langle \sigma_z \rangle = 0 \quad .$$

I leave it up to you to square this with the commutation relations of the Pauli matrices with the Hamiltonian, and how to visualize this as a precession in real and Bloch-sphere space.

[P_{II}] Problem list

Do not get frightened by the following problems. They do not go beyond the concepts developed thus far, but their context is different and uses jargon from quantum computing. To solve the problems, the meaning and interpretation of the states and operators is not necessary, but you will gain more if you follow those links below which I inserted for further reading.

Level 2

P 3. A gate is a Hamiltonian switched on for a time τ . Qubit basis $|0\rangle \equiv |z+\rangle$, $|1\rangle \equiv |z-\rangle$; two qubits $|ab\rangle = |a\rangle \otimes |b\rangle$, the first one the control. The gates (do not confuse these U with the time-evolution operator)

$$U_{\text{NOT}} = \sigma_x, \quad U_{\text{Had}} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z), \quad U_{\text{CNOT}} = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes \sigma_x$$

are to be read as $U(\tau) = e^{-iH\tau/\hbar}$ for some Hermitian H , up to a global phase.

- (a) For any Hermitian operator A with $A^2 = \mathbf{1}$, show $e^{-i\alpha A} = \cos \alpha \mathbf{1} - i \sin \alpha A$.
- (b) Let $H = \frac{\hbar\Omega}{2}\sigma_x$. Find the smallest τ with $U(\tau) = U_{\text{NOT}}$ up to a phase, and give that phase. Which state (see Week I) is $U(\tau/2)|0\rangle$ equal to?
- (c) Read the same H as hopping between two bins $|0\rangle, |1\rangle$ in the form of Eq. (3-4): identify w and the on-site terms v_0, v_1 .
- (d) For $H = \frac{\hbar\Omega}{2} \hat{n} \cdot \vec{\sigma}$, use $\frac{d}{dt}\langle L \rangle = \frac{i}{\hbar}\langle [H, L] \rangle$ and $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ to show $\frac{d}{dt}\langle \vec{\sigma} \rangle = \Omega \hat{n} \times \langle \vec{\sigma} \rangle$. Compare this with what was discussed in class.
- (e) Show $U_{\text{Had}}^2 = \mathbf{1}$ and hence $U_{\text{Had}} = i e^{-i\frac{\pi}{2}U_{\text{Had}}}$. Give H , and describe the Hadamard gate as a precession on the Bloch sphere: axis and angle. Where do $|z\pm\rangle$ go?
- (f) Show $U_{\text{CNOT}} = \mathbf{1} - 2P$ with $P = |1\rangle\langle 1| \otimes |x-\rangle\langle x-|$ a projector, hence $U_{\text{CNOT}} = e^{-i\pi P}$ and

$$H = \frac{\pi\hbar}{4\tau}(\mathbf{1} \otimes \mathbf{1} - \sigma_z \otimes \mathbf{1} - \mathbf{1} \otimes \sigma_x + \sigma_z \otimes \sigma_x).$$

Which of these terms cannot be generated by a field acting on each qubit separately?

- (g) *Extension.* A processor applies gates one after another. Show $e^{-i(A+B)\Delta t} = e^{-iA\Delta t}e^{-iB\Delta t} + \frac{1}{2}[A, B]\Delta t^2 + O(\Delta t^3)$. Split the hopping matrix W of the N -bin line into even and odd bonds, and turn e^{-iWt} into a circuit of two-bin gates of type (b). How does the error scale with the number of steps?

[Nielsen & Chuang (10th anniv. ed.), Eqs. 4.4–4.8 and Ex. 4.2 for (a), (d); §4.7, Thm. 4.3 and Eq. 4.103 for (h).]

Level 3

P 5. The proton that draws the picture. A proton in a field $\mathbf{B}$: $H = -\gamma_p \boldsymbol{\sigma} \cdot \mathbf{B}$, with $\gamma_p/2\pi = 42.58 \text{ MHz T}^{-1}$, $\gamma_p > 0$.

- (a) $\mathbf{B} = B_0 \hat{z}$. Introduce the form $H = \omega \frac{\hbar}{2} \sigma_z$ by correctly identifying ω , sign included. Does the proton precess in the same sense as the electron? You may now find the (Larmor) frequency $f_0 = \omega_0/2\pi$ at 1.5 and 3 T being of the same order as the FM radio band, 87.5–108 MHz.
- (b) Now, one individual proton, measured along $\hat{x}$, yields $\pm\hbar/2$, but some detector which would detect the ensuing signal sees the summed moment of all protons. Take the populations of $|z\pm\rangle$ as Boltzmann, $\propto e^{-E_{\pm}/k_B T}$. Show $(N_+ - N_-)/N = \tanh(\hbar\omega_0/2k_B T)$ and evaluate it at 3 T, 310 K. One mm^3 of water holds 6.7×10^{19} protons; how many are

net aligned? In the frame rotating with the protons take, as given, $H_{\text{rot}} = -\gamma_p B_1 \frac{\hbar}{2} \sigma_x$. Starting from $|z+\rangle$, find the time t_{90} after which the state lies on the equator of the Bloch sphere, and which equator state it is. Evaluate for $B_1 = 5.9 \mu\text{T}$.

- (c) Switch B_1 off. Back in the laboratory, compute $\langle \sigma_x \rangle(t)$, $\langle \sigma_y \rangle(t)$ for the state of (c). This rotating moment induces the signal in the coil.
- (d) Imaging: add a gradient, $B_z = B_0 + Gx$. In the language of the $[C_{II}]$ sheet, which $v_{\sigma\sigma'}(x)$ is this? Show that the signal of a proton density $\rho(x)$, with the carrier at ω_0 removed, is $\int dx \rho(x) e^{\pm i\gamma_p Gx t}$: a Fourier transform at $k = \gamma_p Gt$. At $G = 10 \text{ mT m}^{-1}$, by how many hertz do two points 1 mm apart differ?
- (e) *Extension*. Field inhomogeneity gives each proton its own offset δ , and the summed moment of (d) decays. A π pulse about $\hat{x}$ at time T maps the azimuth $\phi \rightarrow -\phi$. Show that every proton is back in phase at $2T$, whatever its δ . What does the echo not undo?

[Sakurai §2.1, “Spin Precession”; Nielsen & Chuang §7.7, refocusing in §7.7.3; E. L. Hahn, Phys. Rev. **80**, 580 (1950), for the echo.]

$[T_{II}]$ **The I-never-read-textbooks issues**

- Let us milk as much out of the framework developed thus far by considering a one-dimensional line with N sites as the state space. Let us now play good, and furnish the world a particle will experience as it dictates its time evolution.

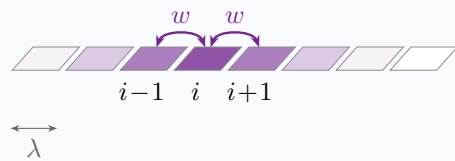
Your only task shall be to conjure an “interesting” Hamiltonian which is subjected only to the constraint of hermiticity.

Interesting questions in this context are to me:

If the spin is a vector operator, can one understand the various bins as possible states of an N -dimensional spin?

Quantum Mechanics I – Week II: Multi-dimensional extension and time evolution

Position, discrete 1D band



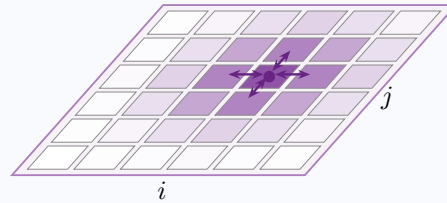
$$\psi_i = \langle i | \Psi \rangle$$

$$\sum_i |\psi_i|^2 = 1$$

$$i \partial_t \psi_i = \sum_{i'} W_{ii'} \psi_{i'}$$

$$W_{i'i} = W_{ii'}^* \quad 2 \text{ neighbours}$$

Position, discrete 2D lattice



$$\psi_{ij} = \langle i, j | \Psi \rangle, \quad |i, j\rangle = |i\rangle \otimes |j\rangle$$

$$\sum_{i,j} |\psi_{ij}|^2 = 1$$

$$i \partial_t \psi_{ij} = \sum_{i',j'} W_{ij,i'j'} \psi_{i'j'}$$

$$4 \text{ neighbours}$$

2-state system (qubit) $\otimes$ Position, discrete 2D lattice

Two 2D grids representing states $\sigma = +$ (blue) and $\sigma = -$ (red). A vertical arrow connects a site in the $\sigma = +$ grid to a site in the $\sigma = -$ grid, labeled $w_{+-} = w_{-+}^*$. The $\sigma = -$ grid has axes x and y .

$$|\Psi\rangle = \sum_{i,j,\sigma} \psi_{ij\sigma} |i, j\rangle \otimes |\sigma\rangle$$

$$\left\{ \begin{array}{c} \psi_{11+} \\ \psi_{12+} \\ \vdots \\ \psi_{ij+} \\ \vdots \end{array} \right\} \quad |+\rangle$$

$$\left\{ \begin{array}{c} \psi_{11-} \\ \psi_{12-} \\ \vdots \\ \psi_{ij-} \\ \vdots \end{array} \right\} \quad |-\rangle$$

4 neighbours + 1 across

$$\lambda \rightarrow 0$$

$$\psi_i \rightarrow \sqrt{\lambda} \psi(x)$$

$$\int dx |\psi|^2 = 1$$

$$i \hbar \partial_t \psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi + v \psi$$

$$\psi_{ij} \rightarrow \lambda \psi(x, y)$$

$$\int dx dy |\psi|^2 = 1$$

$$i \hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + v \psi$$

$$\psi_{ij\sigma} \rightarrow \lambda \psi_\sigma(x, y) = \langle x, y; \sigma | \Psi \rangle$$

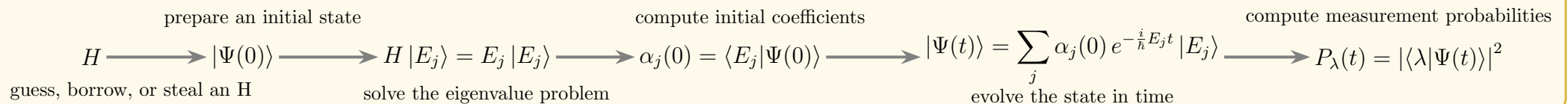
$$\sum_\sigma \int dx dy |\psi_\sigma|^2 = 1$$

$$i \hbar \partial_t \psi_\sigma = -\frac{\hbar^2}{2m} \nabla^2 \psi_\sigma + \sum_{\sigma'} v_{\sigma\sigma'} \psi_{\sigma'}$$

$$\Psi(x, y, t) = \begin{pmatrix} \psi_+(x, y, t) \\ \psi_-(x, y, t) \end{pmatrix}$$

$$v = \omega S_z: \quad e^{-i\omega t/2}, e^{+i\omega t/2}$$

recipe for a Schrödinger ket:



[N_{III}] Uncertainty

[Susskind & Friedman, Lecture 5, §§5.1–5.4]

Complete sets of commuting observables

$$L |\lambda_i, \mu_a\rangle = \lambda_i |\lambda_i, \mu_a\rangle$$

$$M |\lambda_i, \mu_a\rangle = \mu_a |\lambda_i, \mu_a\rangle$$

$$L |\lambda, \mu\rangle = \lambda |\lambda, \mu\rangle, \quad M |\lambda, \mu\rangle = \mu |\lambda, \mu\rangle$$

$$LM |\lambda, \mu\rangle = L\mu |\lambda, \mu\rangle = \lambda\mu |\lambda, \mu\rangle$$

$$LM |\lambda, \mu\rangle = ML |\lambda, \mu\rangle$$

$$[L, M] |\lambda, \mu\rangle = 0$$

$$[L, M] = 0$$

Wave functions

$$|\Psi\rangle = \sum_{a,b,c,\dots} \psi(a, b, c, \dots) |a, b, c, \dots\rangle$$

$$\psi(a, b, c, \dots) = \langle a, b, c, \dots | \Psi \rangle$$

$$P(a, b, c, \dots) = \psi^*(a, b, c, \dots) \psi(a, b, c, \dots)$$

$$\psi(z+) = \langle z+ | \Psi \rangle, \quad \psi(z-) = \langle z- | \Psi \rangle$$

$$\psi(x+) = \langle x+ | \Psi \rangle, \quad \psi(x-) = \langle x- | \Psi \rangle$$

$$\sum_{a,b,c,\dots} \psi^*(a, b, c, \dots) \psi(a, b, c, \dots) = 1$$

$$|\Psi\rangle = \sum_j \alpha_j |\psi_j\rangle$$

Measurement

$$L = \begin{pmatrix} r & w \\ w^* & r' \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$L = a \sigma_x + b \sigma_y + c \sigma_z + d \mathbf{1}, \quad a, b, c, d \in \mathbb{R}$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y$$

The uncertainty principle

The meaning of uncertainty

$$\langle \Psi | A | \Psi \rangle = \sum_a a P(a)$$

$$\bar{A} = A - \langle A \rangle$$

$$\bar{A} = A - \langle A \rangle \mathbf{1}$$

$$\bar{a} = a - \langle A \rangle$$

$$(\Delta A)^2 = \sum_a \bar{a}^2 P(a)$$

$$(\Delta A)^2 = \sum_a (a - \langle A \rangle)^2 P(a)$$

$$(\Delta A)^2 = \langle \Psi | \bar{A}^2 | \Psi \rangle$$

$$(\Delta A)^2 = \langle \Psi | A^2 | \Psi \rangle \quad \text{for } \langle A \rangle = 0$$

$[\beth_{III}]$

$[R_{III}]$ **Reading list**

$[P_{III}]$ **Problem list**

Level 1

Level 2

Level 3

$[T_{III}]$ **The I-never-read-textbooks issues**

$[C_{III}]$ Paulo slide