

Three lectures on effective theories for more than three particles

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As for TOE, I doubt that we will ever know all the laws governing nature (this opinion is based more on intuition than on logic). I believe that any physical theory will have a restricted field of application, and extensions beyond this will require the modification of basic concepts and ideas.

- Andrei Sakharov, *Memoirs*

0.1 The relation between the Efimov effect and the *unnatural* EFT scaling

I will revisit the discussion of the Efimov effect as given in [2] using 3-body integral equations and translate the arguments into the language of the contact effective field theory.

prelude: Let me first recall the identity of non-relativistic quantum field theory and “ordinary” quantum mechanics. Both theories aim to find the time evolution of the state $|\Psi, t\rangle$ of a system, and *both* demand *that* state to obey:

$$i\hbar \frac{\partial}{\partial t} \Psi = \left[\sum_{n=1}^A \left(-\frac{\hbar^2}{2m} \nabla_n^2 + V(x_n) \right) + \sum_{i<j} V(x_i - x_j) \right] \Psi(t, x_1, \dots, x_A) \quad , \quad (0.1)$$

which is a particular instance of the general Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi, t\rangle = H(\psi_{\mathbf{x}}^\dagger, \psi_{\mathbf{x}}) |\Psi, t\rangle \quad , \quad (0.2)$$

where the dependence of the Hamiltonian on the field operators is made explicit to to prepare for the field-theoretic EFT formulation. While (0.1) is an equation for the functions Ψ in coordinate representation, (0.2) is kept in the abstract. The latter yields the former by substituting Ψ , i.e., the amplitude to find the A particles at the implied $(3A + 1)$ -dimensional space-time point, which, in the Schrödinger picture ^①, encodes the time dependence by

$$|\Psi, t\rangle = \int_{x_1 \dots x_A} \Psi(t, x_1, \dots, x_A) \psi_{x_1}^\dagger \dots \psi_{x_A}^\dagger |0\rangle \quad , \quad (0.3)$$

and by usage of the Hamiltonian

$$\begin{aligned} H &= \int_x \left[-\frac{\hbar^2}{2m} (\nabla \psi_x^\dagger) (\nabla \psi_x) + V(x) (\psi_x^\dagger \psi_x) + \frac{1}{2} \int_y \psi_x^\dagger \psi_y^\dagger V(x-y) \psi_y \psi_x \right] \\ &\stackrel{\text{eccel!}}{\neq} \int_x \psi_x^\dagger \left(-\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi_x + \frac{1}{2} \int_{x,y} \psi_x^\dagger \psi_y^\dagger V(x-y) \psi_y \psi_x \quad . \end{aligned} \quad (0.4)$$

ex1

Show that the general time-evolution equation (0.2) obtains with the Hamiltonian (0.4) and and the wave function expanded as in (0.3), the specific form (0.1). As technical as this might appear, it forces you to work with the field operators and their commutation relations in a rather unfamiliar context.

To build an eft by considering all symmetry-allowed operator combinations in a Hamiltonian like (0.4), and to order that infinite number of terms based on their operator dimension, is unusual.

ex

Show that the kinetic terms in (0.4) and (2.9) in [6] are equivalent. You might want to start with the interaction-picture field operators

$$\psi_{\mathbf{x}}(t) = e^{iH_0 t} \psi_{\mathbf{x}} e^{-iH_0 t} \quad . \quad (0.5)$$

Use $e^{iH_0 t} a_{\mathbf{p}}^\dagger e^{-iH_0 t} = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t}$ and possibly the momentum representation of the field operators $\psi_{\mathbf{x}}^\dagger = \int_{\mathbf{p}} a_{\mathbf{p}}^\dagger e^{i\mathbf{p} \cdot \mathbf{x}}$.

To recap, I use [7], in which the transition from

$$L = \int_x \left(\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \right) \quad (0.6)$$

^①Field operators $\psi_{\mathbf{x}}^\dagger$ are thus time independent. This keeps confusing me, when the Lagrangean is taken as the defining density because only for a time-dependent Lagrangean, the temporal integration for obtaining the action is sensible.

to the Hamiltonian

$$H = \int_{\mathbf{p}} E_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \quad (0.7)$$

spells out the necessary steps in case of the relativistic scalar field. (*ecce* : if you are never confused when and where things are time dependent, and would like to adhere to the formulation in [6], skip this exercise.)

With the Hamiltonian (0.4), I have partially defined the EFT by specifying its degrees of freedom and a leading order interaction in addition to the free kinetic part (some of the symmetries are implied). However, these choices become *useful* only in combination with the specific process=amplitude=S-matrix of interest (e.g., for the contact interaction $V(x-y) = c \cdot \delta(x-y)$, (0.4) is quite useless if the interest is the transition amplitude of an incoming neutron-proton S-wave state into an outgoing D-wave state).

ex2

Write out the explicit form of neutron-proton states in a specific partial wave of the relative orbital motion. Make reasonable choices for spin, isospin, and coordinate systems. Start from the general expression in (0.3).

prose: It is crucial to understand the EFT program as nothing but the *controlled* expansion of an observable. The qualifier “controlled” implies the existence of small, dimensionless parameters \mathbf{N}_i and a practically computable expansion of the observable in a power series of these parameters which has converging character. So having specified degrees of freedom in form of field operators (“What do we describe?”), and symmetries which constrain the Hamiltonian (“How does that *what* interact with itself and what types of encounters with other objects in space-time respect its given bundle structure group?”), it “remains” to identify the small parameters (“For a given process, what are the relevant encounters, i.e., type and frequency?”).

detour: The S-matrix is the time evolution operator, $\exp(-iHt)$, in the limit of sufficiently large times:

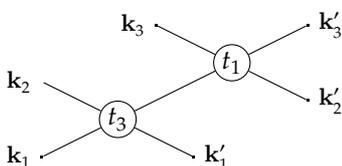
$$\langle \mathbf{p}_1 \dots \mathbf{p}_A | S | \mathbf{p}'_1 \dots \mathbf{p}'_A \rangle = \lim_{t \rightarrow \infty} \langle \mathbf{p}_1 \dots \mathbf{p}_A | \exp[-iH(2t)] | \mathbf{p}'_1 \dots \mathbf{p}'_A \rangle \quad (0.8)$$

The field-theoretical Hamiltonian is as in (0.4), i.e., field operators given in the Schrödinger picture.

3-body amplitudes: I want to employ the above approach to the non-relativistic contact EFT for the description of three particles. Thereby, I follow [3, 4]. Here—as everywhere—the object of interest is an S-matrix element. Specifically, consider the transition from an initial state of three noninteracting particles with momenta $\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3$ into a final state with momenta $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ with an interaction active only at finite times/distances which is the simplest non-trivial case as investigated in the earlier lectures. In the approach chosen here, it assumes the form:

$$H = H_0 + V \quad (0.9)$$

$$\begin{aligned} \langle \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | S | \mathbf{p}'_1 \mathbf{p}'_2 \mathbf{p}'_3 \rangle &= \lim_{t \rightarrow \infty} \langle \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | \exp[-iH(2t)] | \mathbf{p}'_1 \mathbf{p}'_2 \mathbf{p}'_3 \rangle \\ &= \lim_{t \rightarrow \infty} \langle \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | \exp[-i(H_0 + V)(2t)] | \mathbf{p}'_1 \mathbf{p}'_2 \mathbf{p}'_3 \rangle \end{aligned} \quad (0.10)$$



ex

Obtain

0.2 The Amado-Greenwood theorem

I will present the theorem of Amado and Greenwood [1] and explain its relevance for the power counting in effective field theories, more specifically, its implications for potential unnatural scaling of many-body forces beyond the momentum-independent three-body term.

The state $|\mathbf{pk}\rangle$ is a 3-body state with particles ijk having momenta \mathbf{p} , \mathbf{k} , and $-(\mathbf{p} + \mathbf{k})$.

$$\psi(\mathbf{pk}) = \int d^3p' d^3k' K(\mathbf{pk}|\mathbf{p}'\mathbf{k}') \psi(\mathbf{p}'\mathbf{k}') \quad . \quad (\text{homogeneous LS})$$

$$K(\mathbf{pk}|\mathbf{p}'\mathbf{k}') = \frac{2 \langle \mathbf{k} + \frac{1}{2}\mathbf{p} | t(E - \frac{3}{4}\mathbf{p}^2) | \mathbf{p}' + \frac{1}{2}\mathbf{k}' \rangle}{E - \mathbf{k}^2 - \mathbf{p}^2 - \mathbf{k} \cdot \mathbf{p}} \delta(\mathbf{k}' - \mathbf{p}) \quad . \quad (\text{3-body kernel})$$

$$\langle \mathbf{k} | t(E) | \mathbf{k}' \rangle = u(\mathbf{k}) \tau(E) u(\mathbf{k}') \quad \text{with} \quad \tau(E) = \frac{-1}{g^{-1} + \int d^3q \frac{u^2(\mathbf{q})}{E - \mathbf{q}^2}} \stackrel{E \rightarrow 0}{=} -\frac{1}{2\pi^2} \left(\frac{1}{a} + \sqrt{-E} \right)^{-1} \quad (\text{2-body t-matrix})$$

Compact kernels have

1. a discrete spectrum of eigenvalues,
2. zero as the sole eigenvalue-accumulation point, and
3. they may be approximated by finite-rank operators.

0.3 A conjecture of unnaturality for many-body vertices

A conjecture on the scaling of momentum-independent A -body vertices based on the empirically observed ([5]) promotion of a 4-body counterterm at next to leading order is given.

0.4 The analytic structure of the 2-nucleon S/T -matrices

The analytic structure of S/T -matrices is discussed as it changes in the course of its EFT expansion. In this part, I will focus exclusively on the 2-body sector.

0.5 Understanding the pole structure of S/T -matrices for more than two particles

The canonical visualization of the analytic structure of few-body amplitudes with multiple Riemann sheets is presented. Subsequently, I will discuss the need to move poles of these amplitudes onto different sheets by means of perturbative corrections in the EFT expansion.

0.6 Perturbative corrections to S/T -matrix poles at higher EFT orders

Problems related to and erroneous characterization of states/poles at given EFT order, e.g., physical bound states are predicted as resonances, are discussed. I will present a few methods addressing this issue but also elaborate on the fallacy which infers from such a mischaracterization a deficient renormalization of the theory.

0.7 Grand challenges

1. Prediction of the critical number to sustain a self-bound Brunnian cluster;
2. Explanation, refinement, and/or refutation of the empirically observed analytic structure of few-boson amplitudes close to unitarity;
3. Systematic transition/matching between effective field theories formulated with different degrees of freedom;
4. Reliable precision in multi-channel reaction calculations;

0.8 An incomplete list of pertinent numerical methods

- basis-expansion methods;
- discretization methods;
- Greens-function/propagation methods;
- Faddeev(-Yakubovsky) methods;
- Monte-Carlo methods;
- resonating-group methods;

Nomenclature

$|\mathbf{p}\rangle$ Eigenstate of the momentum operator with eigenvalue \mathbf{p} . The state corresponds to a single particle which can be created from the non-interacting vacuum via $a_{\mathbf{p}}^\dagger |0\rangle = |\mathbf{p}\rangle$.

$\psi_{\mathbf{x}/x/l}^{(\dagger)}$ non-relativistic field annihilation (creation) operator

$$\psi_{\mathbf{x}}^{(\dagger)} = \int_{\mathbf{p}} (2\omega_{\mathbf{p}})^{-1/2} a_{\mathbf{p}}^{(\dagger)} e^{(-)i\mathbf{p}\cdot\mathbf{x}} \quad (\text{Schrödinger picture})$$

$$\psi_x^{(\dagger)} = e^{iHt} \psi_{\mathbf{x}}^{(\dagger)} e^{-iHt} \quad (\text{Heisenberg picture})$$

$$\psi_l^{(\dagger)} = \int_{\mathbf{p}} (2\omega_{\mathbf{p}})^{-1/2} a_{\mathbf{p}}^{(\dagger)} e^{(-)ip\cdot x} \quad (\text{Interaction picture})$$

$|0\rangle$ Ground state (“vacuum”) of universe. This state is an element of the Fock space: $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ with \mathcal{H}_n the Hilbert space of n particles. The latter is the tensor product $\mathcal{H}_n = \bigotimes_{i=1}^n \mathcal{H}_1^{(i)}$ of those single-particle Hilbert spaces which comprise the composite system of n , possibly different, particles.

T time-ordering operator

$$\langle 0 | T(\psi_x \psi_y) | 0 \rangle \equiv \theta(x^0 - y^0) \langle 0 | \psi_x \psi_y | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \psi_y \psi_x | 0 \rangle$$

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